ON THE EXISTENCE OF NONCOMPACT BOUNDED LINEAR OPERATORS BETWEEN CERTAIN BANACH SPACES

BY

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ABSTRACT

It is proved that if the Banach space E has an unconditional basis and if F is another Banach space, the following two assertions are equivalent: (1) There is a non-compact bounded linear operator from E into F'. (2) The space of bounded linear operators from E into F' has a subspace isomorphic to c_0.

Let F be a Banach space. This paper determines a necessary and sufficient condition in order that $K(E,F')$ be a proper subspace of $L(E,F')$, for the case where E is a Banach space with unconditional basis. When E' or F' have the approximation property, this gives a condition when $E' \otimes_\lambda F' \neq (E \otimes_\gamma F)'$. Here, γ denotes the greatest crossnorm and λ the least crossnorm as defined in Schatten [4]. For the case where $E = F = l_2$, it is well known that $(E \otimes_\gamma F)' = L(E,F') \neq E' \otimes_\lambda F'$ and that the subspace in $L(E,F')$ consisting of all bounded linear operators given by diagonal matrices is isometric to l_∞. The necessary and sufficient condition (see (1.5) which we give in order that $K(E,F') \neq L(E,F')$ is to require that $L(E,F')$ contain a complemented subspace which is norm isomorphic to l_∞ or that $E \otimes_\lambda F$ contain a complemented subspace which is norm isomorphic to l_1.

Whenever we refer to a Banach space E as having a Schauder basis we shall assume that this Schauder basis has been normalized, i.e. $\| \pi_n \| = 1$ for all n, where $\pi_n(x) = \sum_{1 \leqq i \leqq n} (x, e'_i) e_i$ and $\{e_i, e'_i\}$ is the Schauder basis system. (See Theorem 1, p. 67 of Day [3].)

* This research was supported in part by National Science Foundation Grants GP 12027 and GU 3171.

The author thanks the referee for simplifying the proof.

Received December 22, 1970 and in revised forms April 6, 1971 and September 14, 1971

If E, F are arbitrary Banach spaces, $L(E,F)$ will denote the space of bounded linear operators from E into F, given the operator norm; $K(E,F)$ will denote the subspace of compact operators.

1. Definition 1.1. Let E be a Banach space with Schauder basis $\{e_i, e_i'\}$. We say that $\{e_i, e_i'\}$ is an unconditional basis if, in addition, $\Sigma_i(x, e_i')e_i$ converges (in the norm) to x regardless of rearrangement.

If E is a Banach space whose elements are sequences $x = (x(1), \cdots x(i), \cdots)$ (where $x(i)$ is a scalar), and S is any set of indices then $\pi_S(x)$ denotes the sequence whose ith term is $x(i)$ if $i \in S$ and is zero if otherwise. If E is a Banach space with unconditional basis then it may be regarded as a space whose elements are sequences and π_S is a projection of E into itself; without losing generality, we may assume that E is normed in such a way that $\| \pi_S \| = 1$ (see Theorem 1, p. 73 of Day [3].) If $S = \{1, 2, \cdots, n\}$, we shall write π_n to denote π_S. Let $n(1) < n(2) < \cdots < n(k) < \cdots$ be a sequence of indices. Let $S(k) = \{i : n(k-1) + 1 \leqq i \leqq n(k)\}$. We shall write $\pi_{[k]}$ to denote $\pi_{S(k)}$.

Proposition 1.2. *Let E be a Banach space with a Schauder basis. Let F be a Banach space. Let $T: E \to F$ be a bounded linear operator. If T is not compact, then there exists a sequence of indices $n(1) < n(2) < \cdots < n(k) < \cdots$ and an $\varepsilon > 0$ so that $\| T\pi_{[k]} \| > \varepsilon$ for all k.*

Proof. For, if the conclusion were false then we shall have that $\lim_n \| T\pi_n - T \| = 0$, so that T is compact, contrary to assumption.

Theorem 1.3. (Bessaga–Pełczyński). *Let F be a Banach space. Let $\Sigma_i\, y_i$ be an unconditionally summable series in the weak topology of F. Suppose that $\Sigma_i\, y_i$ does not converge in the norm. Then there is a sequence of indices $n(1) < n(2) < \cdots < n(k) < \cdots$ so that $\{Y_k\}$ forms a Schauder basis for a subspace which is norm isomorphic to c, where $Y_k = \Sigma_{n(k-1) \leqq i \leqq n(k)}\, y_i$. Furthermore, if F is also the dual space of some Banach space, say $F = G'$, then* (a) *the weak* limit of $\Sigma_k c_k Y_k$ exists whenever $\{c_k\}$ is a bounded sequence of scalars and the subspace in F of all such elements forms a complemented subspace in F which is norm isomorphic to l_∞;* (b) *G contains a complemented copy of l_1.*

Proof. See Theorem 5 in Bessaga–Pełczyński [1] and Theorem 1 in Bessaga–Pełczyński [2].

The following is a well known consequence:

COROLLARY 1.4. *Let $T: c_0 \to F$ be a non-compact bounded linear operator. Then*

(a) *F contains a subspace which is norm isomorphic to c_0.*

(b) *If F_1, F_2 are Banach spaces and $F_1 \times F_2$ contains a subspace which is norm isomorphic to c_0, then either F_1 or F_2 contains a subspace which is norm isomorphic to c_0.*

Let E and F denote infinite dimensional spaces.

THEOREM 1.5. *Let E be a Banach space with unconditional basis. Let F be an arbitrary Banach space. Then the following are equivalent:*

(1) *$K(E,F')$ is a proper subspace of $L(E,F') \cong (E \otimes_\gamma F)'$.*

(2) *$(E \otimes_\gamma F)'$ contains a subspace which is norm isomorphic to c_0 (equivalently, a complemented subspace which is norm isomorphic to l_∞).*

(3) *$E \otimes_\gamma F$ contains a complemented subspace which is norm isomorphic to l_1; Moreover, in the special case where $E = c_0$, we may add to the above list of equivalent conditions:*

(4) *F' contains a complemented subspace which is norm isomorphic to l_∞.*

(5) *F contains a complemented subspace which is norm isomorphic to l_1.*

PROOF. The assertion of (1) implies that there is a non-compact bounded linear operator $T: E \to F'$. By (1.2), we may conclude that there exists $\varepsilon > 0$ and a sequence of indices $n(1) < n(2) < \cdots < n(k) < \cdots$ so that $\| T\pi_{[k]} \| > \varepsilon$ for all k. Let σ be a finite index set. Let $\pi_{[\sigma]} = \sum_{k \in \sigma} \pi_{[k]}$. Let $T_k = T\pi_{[k]}$. Then $\sum_{k \in \sigma} T_k = T\pi_{[\sigma]}$. Hence, $\| \sum_{k \in \sigma} T_k \| \leqq \| T \|$ for any finite index set σ. Thus, $\sum_k T_k$ is an unconditionally summable series in the weak topology of $L(E,F')$. Since $\| T_k \| = \| T\pi_{[k]} \| > \varepsilon$, this series is not Cauchy in the norm. Therefore (1.3) applies and $L(E,F')$ contains a subspace which is norm isomorphic to c_0.

(2) $\Rightarrow$ (3): immediate from (1.3).

(3) $\Rightarrow$ (1): If $E \otimes_\gamma F$ has a complemented subspace which is norm isomorphic to l_1, then $(E \otimes_\gamma F)'$ contains a complemented subspace which is norm isomorphic to l_∞. It suffices, then, to show that if $L(E,F') \cong (E \otimes_\gamma F)'$ contains a subspace which is norm isomorphic to c_0, then $K(E,F')$ is a proper subspace of $L(E,F')$.

First, assume that F' contains a subspace which is norm isomorphic to c_0. In this case, (1.3) shows that F' contains a complemented subspace which is norm isomorphic to l_∞. But since l_∞ contains an isomorphic copy of any separable Banach space, F' contains an isomorphic copy of E, and therefore $K(E, F') \neq L(E,F')$.

Suppose next that F' does not contain a subspace which is norm isomorphic to c_0 and that: $K(E,F') = L(E,F')$. Since every seperable Banach space is a quotient space of l_1 it follows that E does not contain a complemented subspace which is norm isomorphic to l_1. By assumption, there exist norm one operators $\{T_i\}$ in $L(E,F')$ which forms a Schauder basis for a subspace C which is norm isomorphic to c_0. Thus, $\| \sum_{i\in\sigma} T_i \| \leqq K$ for any finite index set σ. The operators $\{T_i\}$ are all compact operators. Define $F'_k = F'$ for all k and $p_{m,n}: C \to \times_{m \leqq k \leqq n} F'_k$ by setting $p_{m,n}(T_i) = (T_i(e_m), \cdots T_i(e_n))$ where $\{e_j\}$ is the unconditional basis of E.

If any of the operators $p_{m,n}$ were non-compact then (1.4) shows that at least one of the spaces $F'_k = F'$ contains a subspace which is norm isomorphic to c_0, contrary to assumption. Thus all the operators $p_{m,n}$ are compact and so $\lim_i \| p_{m,n}(T_i) \| = 0$ since T_i is a weakly o-convergent sequence. Since E does not contain a complemented subspace which is norm isomorphic to l_1, we may assert that $\lim_n \| T_k\pi_n - T_k \| = 0$ for each k. (For, by Theorem 3 p. 76 of Day [3], E' has an unconditional basis. By Schauder's Theorem, the adjoint operator T'_k is compact and hence $\lim_n \| \pi'_n T'_k - T'_k \| = 0$. Therefore, $\lim_n \| T_k\pi_n - T_k \| = 0$.)

Now choose a subsequence $\{T_k\}$ of $\{T_i\}$ and choose indices $n(k)$ inductively so that:

$$\| T_k\pi_{n(k)} - T_k \| \leqq (\tfrac{1}{2})^k \tag{1.5.1}$$

$$\| T_k\pi_{n(k-1)} \| \leqq (\tfrac{1}{2})^k \tag{1.5.2}$$

Thus, if $\pi_{[k]} = \pi_{n(k)} - \pi_{n(k-1)}$, then $\| T_k\pi_{[k]} - T_k \| < (\tfrac{1}{2})^{k-1}$.

Therefore, $\sum_k T_k\pi_{[k]}$ is an unconditionally summable series in the weak topology which is not norm convergent. By (1.3), the weak* limit $T = \sum_k T_k\pi_{[k]}$ is an operator in $L(E,F')$. If T is a non-compact operator, then we are finished. Suppose then that T is compact. Let F_0 denote the closed linear subspace generated by

$$\{T_k(E): k = 1, 2, \cdots\}.$$

Since each T_k is a compact map, F_0 is a separable Banach space and hence (by Theorem 9, p. 185 of Banach: Théorie des Opérations Linéaires; Chelsea Publishing Co., New York) F_0 is norm isomorphic to a suitable subspace of a Banach space G with Schauder basis system $\{g'_i, g_i\}$. Without explicitly referring to the isomorphism mapping F_0 into G, we shall hereafter regard T and T_k as mapping E into G. Since T is compact, there exists an integer p so that

$$\| T - \pi_p T \| < \tfrac{1}{4}, \text{ where } \pi_p(z) = \sum_{1 \leq i \leq p} (z, g_i')g_i.$$

If x_k are unit vectors in E chosen (by using (1.5.1)) so that $\| T_k\pi_{[k]}(x_k) \| > \frac{1}{2}$, then, using the the fact that $T_k\pi_{[k]} = T\pi_{[k]}$, we get:

$$\| \pi_p T_k\pi_{[k]}(x_k) \| > \| T_k\pi_{[k]}(x_k) \| - \| (T_k\pi_{[k]} - \pi_p T_k\pi_{[k]}) (x_k) \| =$$

$$\| T_k\pi_{[k]}(x_k) \| - \| (1 - \pi_p)T_k\pi_{[k]}(x_k) \| > \tfrac{1}{2} - \| (1 - \pi_p)T\pi_{[k]}(x_k) \| > \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4} .$$

Since $\pi_p(G)$ is finite dimensional, there is a subsequence of $\{\pi_p T_k\pi_{[k]}(x_k)\}$ which converges in the norm to some non-zero vector z where $\| z \| \geq \frac{1}{4}$. We continue to denote this subsequence by $\{\pi_p T_k\pi_{[k]}(x_k)\}$. Let z' be a linear functional of norm one so that $z'(z) = \| z \|$. Since

$$\|\pi_p \sum_{k \in \sigma} T_k\pi_{[k]}\| = \| \sum_{k \in \sigma} \pi_p T_k\pi_{[k]}\| \leqq \| \pi_p \| \ (K + 1)$$

we have that:

$$\| \sum_{k \in \sigma} (\pi_p T_k\pi_{[k]}(\,\cdot\,), z')\| \leqq \| \pi_p \| \ (K + 1) \| z' \|$$

for any finite index set σ.

Thus, $\Sigma_k(\pi_p T_k\pi_{[k]}(\,\cdot\,), z')$ is weakly unconditionally summable in E' but is not norm convergent since $(\pi_p T_k\pi_{[k]}(x_k), z')$ converges to $\| z \|$. By part (b) of Theorem (1.3), we see that E contains a complemented subspace isomorphic to l_1, contrary to assumption. Thus, T has to be non-compact.

Q.E.D.

(1) $\Rightarrow$ (4): If $E = c_0$ and $K(E, F')$ is a proper subspace of $(E \otimes_\gamma F)'$ then the argument above implies the existence of a non-compact operator $T: E \to F'$. By (1.4) and (1.3) F' contains a complemented subspace which is norm isomorphic to l_∞.

(4) $\Rightarrow$ (1) is obvious and (4) $\Leftrightarrow$ (5) is immediate from (1.3)

REMARK. If either E' or F' has the approximation property, then the above theorem may be replaced as follows: Let E be a Banach space with unconditional basis.Then, $E' \otimes_\lambda F'$ is a proper subspace of $(E \otimes_\gamma F)'$ if and only if $E \otimes_\gamma F$ contains a complemented subspace which is norm isomorphic to l_1.

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